Math Sci (2017) 11:79–86
DOI 10.1007/s40096-017-0209-1

ORIGINAL RESEARCH

Some convergence results for nearly asymptotically nonexpansive nonself mappings in CAT(κ) spaces

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Received: 22 October 2015 / Accepted: 23 January 2017 / Published online: 7 February 2017

Abstract The purpose of this paper is to prove the demiclosedness principle and convergence theorems for nearly asymptotically nonexpansive nonself mappings in CAT(κ) spaces with $\kappa > 0$. Our results extend and improve some recent results announced in the current literature.

Keywords Fixed point · Nearly asymptotically nonexpansive nonself mapping · Demiclosedness principle · $\triangle$-convergence · Strong convergence · CAT(κ) space

Mathematics Subject Classification 47H10 · 54E40

Introduction

Throughout this paper, $\mathbb{N}$ is the set of all positive integers and $\mathbb{R}$ is the set of all real numbers. Let K be a nonempty subset of a metric space (X, d) and $T : K \to X$ be a nonself mapping. Denote by $F(T) = \{x \in K : Tx = x\}$, the set of fixed points of T. A nonself mapping T is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in K.$$

A subset K of X is said to be a *retract* of X if there exists a continuous mapping $P : X \to K$ such that $Px = x$ for all $x \in K$. A mapping $P : X \to K$ is said to be a *retraction* if $P^2 = P$. It follows that if P is a retraction, then $Py = y$ for all y in the range of P.

Definition 1 [11] Let K be a nonempty subset of a metric space (X, d) and P be a nonexpansive retraction of X onto K. A nonself mapping $T : K \to X$ is said to be

(i) *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive number k_n such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x, y), \quad \forall x, y \in K;$$

(ii) *uniformly L-Lipschitzian* if $k_n = L$ for all $n \in \mathbb{N}$;

(iii) *asymptotically nonexpansive* if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$.

The class of nearly Lipschitzian nonself mappings is an important generalization of the class of Lipschitzian nonself mappings and was introduced by Khan [19].

Definition 2 [19] Let K be a nonempty subset of a metric space (X, d), P be a nonexpansive retraction of X onto K and fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A nonself mapping $T : K \to X$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n(d(x, y) + a_n), \quad \forall x, y \in K. \tag{1}$$

The infimum of constants k_n satisfying (1) is denoted by $\eta(T(PT)^{n-1})$ and is called *nearly Lipschitz constant*.

Remark 1 [19] For $n = 1$, the inequality (1) can be written as:

$$d(T(PT)^{1-1}x, T(PT)^{1-1}y) \leq k_1(d(x, y) + a_1),$$

where we have to take a_1 as zero. Thus in this case, we have

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$d(T(PT)^{1-1}x, T(PT)^{1-1}y) \le k_1 d(x,y)$.

Definition 3 [19] A nearly Lipschitzian nonself mapping T with the sequence $\{a_n, \eta(T(PT)^{n-1})\}$ is said to be *nearly asymptotically nonexpansive* if $\eta(T(PT)^{n-1}) \ge 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T(PT)^{n-1}) = 1$.

Agarwal et al. [2] introduced the modified S-iteration process in a Banach space:

$$\begin{cases} x_1 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1]. This iteration is independent of those of modified Mann iteration in [31] and modified Ishikawa iteration in [35] and reduces to S-iteration of Agarwal et al. [2] when $T^n = T$ for all $n \in \mathbb{N}$. The convergence of S-iteration for different classes of mappings in different spaces has been studied by many authors (see, e.g., [3–5, 17, 18, 20]).

In this paper, we prove the demiclosedness principle for nearly asymptotically nonexpansive nonself mappings in $CAT(\kappa)$ spaces. Also, we present the strong and Δ-convergence theorems of the modified S-iteration process for mappings of this type in a $CAT(\kappa)$ space. Our results extend and improve the corresponding results of Khan [19], Saluja et al. [30], Khan and Abbas [20] and many other results in this direction.

Preliminaries on CAT(κ) space

For a real number κ, a CAT(κ) space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding comparison triangle in a model space with the curvature κ. The term 'CAT(κ)' was coined by Gromov [16, p.119] and the initials are in honor of Cartan, Alexandrov and Toponogov, each of whom considered similar conditions in varying degrees of generality. Fixed point theory in CAT(κ) spaces was first studied by Kirk [21, 22]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see, e.g., [1, 9, 10, 12, 13, 15, 23, 24, 27, 29, 30, 33, 34]). Since any CAT(κ) space is a CAT(κ') space for $\kappa' \ge \kappa$ (see [6, p. 165]), all results for a CAT(0) space can immediately be applied to any CAT(κ) space with $\kappa \le 0$.

Let (X, d) be a metric space and let $x, \ y \in X$ with $d(x,y) = l$. A *geodesic path* joining x to y (or, more briefly, a *geodesic* from x to y) is an isometry $c : [0,l] \subset \mathbb{R} \to X$ such that $c(0) = x$ and $c(l) = y$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. A geodesic segment joining x and y is not necessarily unique in general. When it is unique, this geodesic segment is denoted by $[x, y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that $d(x,z) = \alpha d(x,y)$ and $d(y,z) = (1-\alpha)d(x,y)$. In this case, we write $z = (1-\alpha)x \oplus \alpha y$ for simplicity.

The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be a *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x, y \in X$. Let $D \in (0,\infty]$. If for every x, $y \in X$ with $d(x,y) < D$, a geodesic from x to y exists, then X is said to be *D-geodesic space*. Moreover, if such a geodesic is unique for each pair of points then X is said to be a *D-uniquely geodesic*. Notice that X is a geodesic space if and only if it is a D- geodesic space.

A subset K of X is said to be *convex* if K includes every geodesic segment joining any two of its points. The set K is said to be *bounded* if $\operatorname{diam}(K) = \sup\{d(x,y) : x, y \in K\} < \infty$.

To define a CAT(κ) space, we use the following concept called *model space*. For $\kappa = 0$, the two-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the two-dimensional sphere $(\frac{1}{\sqrt{\kappa}})\mathbb{S}^2$ whose metric is a length of a minimal great arc joining each of the two points. For $\kappa < 0$, M_κ^2 is the two-dimensional hyperbolic space $(\frac{1}{\sqrt{-\kappa}})\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance.

The diameter of M_κ^2 is denoted by

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & \kappa > 0, \\ +\infty & \kappa \le 0. \end{cases}$$

A *geodesic triangle* $\triangle(x,y,z)$ in a metric space (X, d) consists of three points x, y, z in X (the vertices of $\triangle$) and three geodesic segments between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x,y,z)$ in (X, d) is a triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$d(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), d(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}) \quad \text{and} \quad d(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x})$$

(see [6, Lemma 2.14]). If $\kappa \le 0$, then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$, such a comparison triangle exists whenever $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

A geodesic triangle $\triangle(x,y,z)$ in X is said to satisfy the CAT(κ) *inequality* if for any $p, q \in \triangle(x,y,z)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x},\bar{y},\bar{z})$, one has

$$d(p,q) \le d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Now, we are ready to introduce the concept of CAT(κ) space in the following definition taken from [6].

Definition 4

(i) If $\kappa \le 0$, then a metric space (X, d) is called a CAT(κ) *space* if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

(ii) If $\kappa > 0$, then a metric space (X, d) is called a CAT(κ) *space* if it is D_κ-geodesic and any geodesic triangle $\triangle(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Notice that in a CAT(0) space (X, d) if $x, y, z \in X$, then the CAT(0) inequality implies

$$\text{(CN)}\ d^2\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right) \le \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y).$$

This is the *(CN) inequality* of Bruhat and Tits [8]. This inequality is extended by Dhompongsa and Panyanak [14] as

$$(\text{CN}^*) d^2(z, (1-\alpha)x \oplus \alpha y) \le (1-\alpha)d^2(z, x) + \alpha d^2(z, y) - \alpha(1-\alpha)d^2(x, y)$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let $R \in (0, 2]$. Recall that a geodesic space (X, d) is said to be *R-convex* (see [26]) if for any three points $x, y, z \in X$, we have

$$d^2(z, (1-\alpha)x \oplus \alpha y) \le (1-\alpha)d^2(z, x) + \alpha d^2(z, y) - \frac{R}{2}\alpha(1-\alpha)d^2(x, y). \tag{3}$$

It follows from the (CN*) inequality that a CAT(0) space is R-convex for $R = 2$.

The following lemma is a consequence of Proposition 3.1 in [26].

Lemma 1 [27, Lemma 2.3] *Let $\kappa > 0$ and (X, d) be a complete CAT (κ) space with diam $(X) \le \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \pi/2)$. Then, (X, d) is R-convex for $R = (\pi - 2\epsilon)\tan(\epsilon)$.*

In the sequel, we need the following lemma.

Lemma 2 [6, p. 176] *Let $\kappa > 0$ and (X, d) be a complete CAT (κ) space with diam $(X) \le \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \pi/2)$. Then*

$$d((1-\alpha)x \oplus \alpha y, z) \le (1-\alpha)d(x, z) + \alpha d(y, z)$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

We now collect some elementary facts about CAT(κ) spaces. Most of them are proved in the setting of CAT(1) spaces. For completeness, we state the results in a CAT(κ) space with $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

Further, the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known [15, Proposition 4.1] that in a CAT(κ) space X with diam$(X) < \frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point.

Now, we can give the concept of $\triangle$-convergence and collect some of its basic properties.

Definition 5 [23, 25] A sequence $\{x_n\}$ is $\triangle$-*convergent* to $x \in X$ if x is the unique asymptotic center of any subsequence of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x the $\triangle$-limit of $\{x_n\}$.

Lemma 3 *Let $\kappa > 0$ and (X, d) be a complete CAT (κ) space with diam $(X) \le \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \pi/2)$. Then, the following statements hold:*

(i) [15, Corollary 4.4] *Every sequence in X has a $\triangle$-convergent subsequence;*

(ii) [15, Proposition 4.5] *If $\{x_n\} \subseteq X$ and $\triangle\text{-}\lim_{n\to\infty} x_n = x$, then $x \in \cap_{k=1}^{\infty}\overline{conv}\{x_k, x_{k+1}, \ldots\}$, where $\overline{conv}(A) = \cap\{B : B \supseteq A$ and B is closed and convex $\}$.*

By the uniqueness of asymptotic centers, Panyanak [27] obtained the following lemma.

Lemma 4 [27, Lemma 2.7] *Let $\kappa > 0$ and (X, d) be a complete CAT (κ) space with diam $(X) \le \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

The following lemma is crucial in the study of iteration processes in both metric and Banach spaces and it was proved by Qihou [28].

Lemma 5 [28, Lemma 2] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers such that*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Demiclosedness principle

It is well known that one of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's *demiclosedness principle* [7] which states that if K is a nonempty closed convex subset of a uniformly convex

Banach space X and $T: K \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0, that is, for any sequence $\{x_n\}$ in K if $x_n \to x$ weakly and $(I-T)x_n \to 0$ strongly, then $(I-T)x = 0$, where I is the identity mapping of X. Saluja et al. [30] proved the demiclosedness principle for nearly asymptotically nonexpansive self mappings in a CAT(κ) space. Now, we prove the demiclosedness principle for nearly asymptotically nonexpansive nonself mappings in this space.

Theorem 1 *Let $\kappa > 0$ and (X, d) be a complete CAT (κ) space with diam $(X) \leq \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, P be a nonexpansive retraction of X onto K and $T: K \to X$ be a uniformly continuous nearly asymptotically nonexpansive nonself mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle$-$\lim_{n\to\infty} x_n = w$, then $w \in K$ and $Tw = w$.*

Proof By Lemma 3, $w \in K$. Now, we define $\Psi(u) = \limsup_{n\to\infty} d(x_n, u)$ for each $u \in K$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can prove that

$$\lim_{n\to\infty} d(x_n, T(PT)^{m-1}x_n) = 0, \quad \forall m \in \mathbb{N}. \tag{4}$$

In fact, it is obvious that the conclusion is true for $m = 1$. Suppose the conclusion holds for m, now we prove that the conclusion is also true for $m+1$. By the uniform continuity of TP, we have

$$\lim_{n\to\infty} d(T(PT)^{m-1}x_n, T(PT)^m x_n) = 0$$

so that

$$\begin{aligned} d(x_n, T(PT)^m x_n) \leq\, & d(x_n, T(PT)^{m-1}x_n) + d(T(PT)^{m-1}x_n, \\ & \times T(PT)^m x_n) \\ \to\, & 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Equation (4) is proved. This implies that

$$\Psi(u) = \limsup_{n\to\infty} d(T(PT)^{m-1}x_n, u), \quad \text{for each } u \in K \text{ and } m \in \mathbb{N}. \tag{5}$$

In (5), taking $u = T(PT)^{m-1}w$, we have

$$\begin{aligned} \Psi(T(PT)^{m-1}w) &= \limsup_{n\to\infty} d(T(PT)^{m-1}x_n, T(PT)^{m-1}w) \\ &\leq \limsup_{n\to\infty} [\eta(T(PT)^{m-1})(d(x_n, w) + a_m)]. \end{aligned}$$

Hence

$$\limsup_{m\to\infty} \Psi(T(PT)^{m-1}w) \leq \Psi(w). \tag{6}$$

Furthermore, for any $n, m \in \mathbb{N}$, it follows from the inequality (3) with $\alpha = \frac{1}{2}$,

$$\begin{aligned} d^2\left(x_n, \frac{1}{2}w \oplus \frac{1}{2}T(PT)^{m-1}w\right) \leq\, & \frac{1}{2}d^2(x_n, w) + \frac{1}{2}d^2(x_n, T(PT)^{m-1}w) \\ & - \frac{R}{8}d^2(w, T(PT)^{m-1}w). \end{aligned}$$

Since $\triangle$-$\lim_{n\to\infty} x_n = w$, letting $n \to \infty$, we get

$$\begin{aligned} \Psi^2(w) &\leq \Psi^2\left(\frac{1}{2}w \oplus \frac{1}{2}T(PT)^{m-1}w\right) \\ &\leq \frac{1}{2}\Psi^2(w) + \frac{1}{2}\Psi^2(T(PT)^{m-1}w) - \frac{R}{8}d^2(w, T(PT)^{m-1}w), \end{aligned}$$

which yields that

$$d^2(w, T(PT)^{m-1}w) \leq \frac{4}{R}[\Psi^2(T(PT)^{m-1}w) - \Psi^2(w)]. \tag{7}$$

By (6) and (7), we have $\lim_{m\to\infty} d(w, T(PT)^{m-1}w) = 0$. In view of the continuity of TP, we obtain

$$w = \lim_{m\to\infty} T(PT)^m w = \lim_{m\to\infty} TP\left(T(PT)^{m-1}w\right) = TPw = Tw.$$

This completes the proof. □

From Theorem 1, we now derive the following result, yet is new in the literature.

Corollary 1 *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space (X, d), P be a nonexpansive retraction of X onto K and $T: K \to X$ be a uniformly continuous nearly asymptotically nonexpansive nonself mapping. If $\{x_n\}$ is a sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle$-$\lim_{n\to\infty} x_n = w$, then $w \in K$ and $Tw = w$.*

Proof It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (see [6]). Then, (K, d) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa > 0$. Notice also that K is R-convex for $R = 2$. Since K is bounded, we can choose $\epsilon \in (0, \pi/2)$ and $\kappa > 0$ so that diam$(K) \leq \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 1. □

Convergence theorems of the modified S-iteration process

We start with $\triangle$-convergence of the modified S-iterative sequence for nearly asymptotically nonexpansive nonself mappings in CAT(κ) spaces.

Theorem 2 *Let $\kappa > 0$ and (X, d) be a complete CAT (κ) space with diam $(X) \leq \frac{\pi/2-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \pi/2)$. Let*

K be a nonempty closed convex subset of X, P be a nonexpansive retraction of X onto K and $T : K \to X$ be a uniformly continuous nearly asymptotically nonexpansive nonself mapping with the sequence $\{a_n, \eta(T(PT)^{n-1})\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\left(\eta(T(PT)^{n-1}) - 1\right) < \infty$. Let $\{x_n\}$ be a sequence in K defined by

$$\begin{cases} x_1 \in K, \\ y_n = P((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n), \\ x_{n+1} = P((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n), \quad n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0, 1) *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *If $F(T) \neq \emptyset$, then $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof We divide our proof into three steps.

Step 1. First, we prove that

$$\lim_{n\to\infty} d(x_n, p) \text{exists for each } p \in F(T). \tag{9}$$

Let $p \in F(T)$. Since T is a nearly asymptotically nonexpansive nonself mapping, by (8) and Lemma 2, we have

$$\begin{aligned} d(y_n,p) &= d(P((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n),p) \\ &\le d((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n,p) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n d(T(PT)^{n-1}x_n,p) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n \eta(T(PT)^{n-1}) \\ &\quad (d(x_n,p)+a_n) \\ &\le \eta(T(PT)^{n-1})[(1-\beta_n)d(x_n,p) + \beta_n d(x_n,p)] \\ &\quad + \beta_n \eta(T(PT)^{n-1})a_n \\ &\le \eta(T(PT)^{n-1})d(x_n,p) + \eta(T(PT)^{n-1})a_n. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+1},p) &= d(P((1-\alpha_n)T(PT)^{n-1}x_n \\ &\qquad \oplus \alpha_n T(PT)^{n-1}y_n),p) \\ &\le d((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n,p) \\ &\le (1-\alpha_n)d(T(PT)^{n-1}x_n,p) + \alpha_n d(T(PT)^{n-1}y_n,p) \\ &\le \eta(T(PT)^{n-1})[(1-\alpha_n)(d(x_n,p)+a_n) \\ &\quad + \alpha_n(d(y_n,p)+a_n)] \\ &\le \eta(T(PT)^{n-1})[(1-\alpha_n)d(x_n,p) \\ &\quad + \alpha_n \eta(T(PT)^{n-1})d(x_n,p) \\ &\quad + (1+\eta(T(PT)^{n-1}))a_n] \\ &\le \left(\eta(T(PT)^{n-1})\right)^2 d(x_n,p) + [\eta(T(PT)^{n-1}) \\ &\quad + \left(\eta(T(PT)^{n-1})\right)^2]a_n \\ &= (1+\sigma_n)d(x_n,p) + \xi_n, \end{aligned} \tag{10}$$

where $\sigma_n = \left(\eta(T(PT)^{n-1})\right)^2 - 1 = (\eta(T(PT)^{n-1}) + 1)(\eta(T(PT)^{n-1}) - 1)$ and $\xi_n = [\eta(T(PT)^{n-1}) + \left(\eta(T(PT)^{n-1})\right)^2]a_n$. Since $\sum_{n=1}^{\infty}\left(\eta(T(PT)^{n-1}) - 1\right) < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, it follows that $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. Hence, by Lemma 5, we get that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$.

Step 2. Next, we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{11}$$

Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $\{x_n\}, \{y_n\} \subset B(p,R')$ for all $n \in \mathbb{N}$ with $R' < D_k/2$. In view of (3), we have

$$\begin{aligned} d^2(y_n,p) &\le d^2((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n,p) \\ &\le (1-\beta_n)d^2(x_n,p) + \beta_n d^2(T(PT)^{n-1}x_n,p) \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n) \\ &\le (1-\beta_n)d^2(x_n,p) + \beta_n[\eta(T(PT)^{n-1}) \\ &\quad (d(x_n,p)+a_n)]^2 \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n) \\ &\le (\eta(T(PT)^{n-1}))^2 d^2(x_n,p) + Pa_n \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n) \end{aligned} \tag{12}$$

for some $P > 0$. This implies that

$$d^2(y_n,p) \le (\eta(T(PT)^{n-1}))^2 d^2(x_n,p) + Pa_n. \tag{13}$$

From (3) and using (13), we get

$$\begin{aligned} d^2(x_{n+1},p) &\le d^2((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n,p) \\ &\le (1-\alpha_n)d^2(T(PT)^{n-1}x_n,p) + \alpha_n d^2(T(PT)^{n-1}y_n,p) \\ &\quad - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n) \\ &\le (1-\alpha_n)[\eta(T(PT)^{n-1})(d(x_n,p)+a_n)]^2 \\ &\quad + \alpha_n[\eta(T(PT)^{n-1})(d(y_n,p)+a_n)]^2 \\ &\quad - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n) \\ &\le (1-\alpha_n)\left(\eta(T(PT)^{n-1})\right)^2 d^2(x_n,p) + Qa_n \\ &\quad + \alpha_n\left(\eta(T(PT)^{n-1})\right)^2\left[\left(\eta(T(PT)^{n-1})\right)^2 d^2(x_n,p) + Pa_n\right] \\ &\quad + La_n - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n) \\ &\le \left(\eta(T(PT)^{n-1})\right)^4 d^2(x_n,p) + (Q+M+L)a_n \\ &\quad - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n) \end{aligned}$$

$$=\left[1+\left(\eta(T(PT)^{n-1})\right)^{4}-1)\right]d^2(x_n,p)+(Q+M+L)a_n$$
$$-\frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n)$$
$$=[1+(\eta(T(PT)^{n-1})-1)\rho]d^2(x_n,p)+(Q+M+L)a_n$$
$$-\frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n)$$

for some $Q,M,L,\rho>0$. This inequality yields that

$$\frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n)$$
$$\leq d^2(x_n,p)-d^2(x_{n+1},p)+(\eta(T(PT)^{n-1})-1)$$
$$\rho d^2(x_n,p)+(Q+M+L)a_n.$$

Since $\sum_{n=1}^{\infty}a_n<\infty$, $\sum_{n=1}^{\infty}\left(\eta(T(PT)^{n-1})-1\right)<\infty$ and $d(x_n,p)<R'$, we obtain

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n)<\infty.$$

Hence by the fact that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we get

$$\lim_{n\to\infty}d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n)=0. \tag{14}$$

Now, consider (12), we have

$$d^2(y_n,p)\leq[1+((\eta(T(PT)^{n-1}))^2-1)]d^2(x_n,p)+Pa_n$$
$$-\frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n)$$
$$\leq[1+(\eta(T(PT)^{n-1}-1)\mu]d^2(x_n,p)+Pa_n$$
$$-\frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n)$$

for some $\mu>0$. This inequality yields that

$$\frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n)$$
$$\leq d^2(x_n,p)-d^2(y_n,p)+(\eta(T(PT)^{n-1})-1)\mu d^2(x_n,p)+Pa_n.$$

Since $\sum_{n=1}^{\infty}a_n<\infty$, $\sum_{n=1}^{\infty}\left(\eta(T(PT)^{n-1})-1\right)<\infty$, $d(x_n,p)<R'$ and $d(y_n,p)<R'$, we obtain

$$\sum_{n=1}^{\infty}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n)<\infty.$$

Hence by the fact that $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, we have

$$\lim_{n\to\infty}d(x_n,T(PT)^{n-1}x_n)=0. \tag{15}$$

Now using (15), we get

$$d(x_n,y_n)\leq d(x_n,(1-\beta_n)x_n\oplus\beta_nT(PT)^{n-1}x_n)$$
$$\leq\beta_nd(T(PT)^{n-1}x_n,x_n)$$
$$\to0\text{ as }n\to\infty.$$

Also, we observe that

$$\begin{aligned}d(x_{n+1},x_n)&\leq d((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_nT(PT)^{n-1}y_n,x_n)\\&\leq(1-\alpha_n)d(T(PT)^{n-1}x_n,x_n)+\alpha_nd(T(PT)^{n-1}y_n,x_n)\\&\leq(1-\alpha_n)d(T(PT)^{n-1}x_n,x_n)\\&\quad+\alpha_n[d(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)+d(T(PT)^{n-1}x_n,x_n)]\\&=d(T(PT)^{n-1}x_n,x_n)+\alpha_nd(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)\\&\to0\,\text{as}\,n\to\infty.\end{aligned} \tag{16}$$

Therefore, we obtain

$$\begin{aligned}d(x_{n+1},y_n)&\leq d(x_{n+1},x_n)+d(x_n,y_n)\\&\to0\text{ as }n\to\infty.\end{aligned} \tag{17}$$

Furthermore, since

$$d(x_{n+1},T(PT)^{n-1}y_n)\leq d(x_{n+1},x_n)+d(x_n,T(PT)^{n-1}x_n)$$
$$+d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n),$$

using (14)–(16), we have

$$\lim_{n\to\infty}d(x_{n+1},T(PT)^{n-1}y_n)=0. \tag{18}$$

Since every nearly asymptotically nonexpansive mapping is nearly Lipschitzian, then we get

$$\begin{aligned}d(x_n,Tx_n)&\leq d(x_n,T(PT)^{n-1}x_n)+d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_{n-1})\\&\quad+d(T(PT)^{n-1}y_{n-1},Tx_n)\\&=d(x_n,T(PT)^{n-1}x_n)+d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_{n-1})\\&\quad+d(T(PT)^{1-1}(PT)^{n-1}y_{n-1},T(PT)^{1-1}x_n)\\&\leq d(x_n,T(PT)^{n-1}x_n)+d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_{n-1})\\&\quad+k_1d((PT)^{n-1}y_{n-1},x_n)\\&\leq d(x_n,T(PT)^{n-1}x_n)+\eta(T(PT)^{n-1})[d(x_n,y_{n-1})+a_n]\\&\quad+k_1d((PT)^{n-1}y_{n-1},x_n)\\&=d(x_n,T(PT)^{n-1}x_n)+\eta(T(PT)^{n-1})[d(x_n,y_{n-1})+a_n]\\&\quad+k_1d(PT(PT)^{n-1}y_{n-1},Px_n)\\&\leq d(x_n,T(PT)^{n-1}x_n)+\eta(T(PT)^{n-1})[d(x_n,y_{n-1})+a_n]\\&\quad+k_1d(T(PT)^{n-2}y_{n-1},x_n).\end{aligned}$$

Hence (15), (17) and (18) imply that $\lim_{n\to\infty}d(x_n,Tx_n)=0$.

Step 3. Now, we prove that $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.

Let $\omega_W(x_n)=\cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. First, we show that $\omega_W(x_n)\subseteq F(T)$. Let $u\in\omega_W(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By Lemma 3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$-$\lim_{n\to\infty}v_n=v\in K$. Also by (11), we have $\lim_{n\to\infty}d(v_n,Tv_n)=0$. It follows from Theorem 1 that $v\in F(T)$. Moreover, by (9), $\lim_{n\to\infty}d(x_n,v)$ exists. Thus,

$u=v$ by Lemma 4. This implies that $\omega_W(x_n)\subseteq F(T)$. Next, we show that $\omega_W(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and let $A(\{x_n\})=\{x\}$. Since $u\in\omega_W(x_n)\subseteq F(T)$, fsrom (9) $\lim_{n\to\infty} d(x_n,u)$ exists. Again by Lemma 4, $x=u$. Thus, $\omega_W(x_n)=\{x\}$. This means that $\{x_n\}$ is Δ-convergent to a fixed point of T. The proof is completed. □

Next, we discuss the strong convergence of the iterative sequence $\{x_n\}$ defined by (8) for nearly asymptotically nonexpansive nonself mappings in a CAT(κ) space.

Theorem 3 *Let X, K, P, T and $\{x_n\}$ be the same as in Theorem 2. Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$ where $d(x,F(T))=\inf\{d(x,p):p\in F(T)\}$.*

Proof If $\{x_n\}$ converges to $p\in F(T)$, then $\lim_{n\to\infty} d(x_n,p)=0$. Since $0\le d(x_n,F(T))\le d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F(T))=0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F(T))=0$. It follows from (9) that $\lim_{n\to\infty} d(x_n,F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n,F(T))=0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. In fact, it follows from (10) that for any $p\in F(T)$

$$d(x_{n+1},p)\le(1+\sigma_n)d(x_n,p)+\xi_n,\quad \forall n\in\mathbb{N},$$

where $\sum_{n=1}^{\infty}\sigma_n<\infty$ and $\sum_{n=1}^{\infty}\xi_n<\infty$. Hence for any positive integers n, m, we have

$$\begin{aligned}d(x_{n+m},x_n)&\le d(x_{n+m},p)+d(p,x_n)\\&\le(1+\sigma_{n+m-1})d(x_{n+m-1},p)+\xi_{n+m-1}+d(x_n,p).\end{aligned}$$

Since for each $x\ge0$, $1+x\le e^x$, we have

$$\begin{aligned}d(x_{n+m},x_n)&\le e^{\sigma_{n+m-1}}d(x_{n+m-1},p)+\xi_{n+m-1}+d(x_n,p)\\&\le e^{\sigma_{n+m-1}+\sigma_{n+m-2}}d(x_{n+m-2},p)+e^{\sigma_{n+m-1}}\xi_{n+m-2}\\&\quad+\xi_{n+m-1}+d(x_n,p)\\&\le\ldots\\&\le e^{\sum_{i=n}^{n+m-1}\sigma_i}d(x_n,p)+e^{\sum_{i=n+1}^{n+m-1}\sigma_i}\xi_n+e^{\sum_{i=n+2}^{n+m-2}\sigma_i}\xi_{n+1}\\&\quad+\ldots+e^{\sigma_{n+m-1}}\xi_{n+m-2}+\xi_{n+m-1}+d(x_n,p)\\&\le(1+N)d(x_n,p)+N\sum_{i=n}^{n+m-1}\xi_i,\end{aligned}$$

where $N=e^{\sum_{i=1}^{\infty}\sigma_i}<\infty$. Therefore, we have

$$d(x_{n+m},x_n)\le(1+N)d(x_n,F(T))+N\sum_{i=n}^{n+m-1}\xi_i\to0 \text{ as } n,m\to\infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence in K. Since K is a closed subset in a complete CAT(κ) space X, it is complete. We can assume that $\{x_n\}$ converges strongly to some point $p^\star\in K$. As T is continuous, so $F(T)$ is closed subset in K. Since $\lim_{n\to\infty} d(x_n,F(T))=0$, we obtain $p^\star\in F(T)$. This completes the proof. □

Remark 2 In Theorem 3, the condition $\liminf_{n\to\infty} d(x_n,F(T))=0$ may be replaced with $\limsup_{n\to\infty} d(x_n,F(T))=0$.

Recall that a mapping T from a subset K of a metric space (X, d) into itself is *semi-compact* if every bounded sequence $\{x_n\}\subset K$ satisfying $d(x_n,Tx_n)\to0$ as $n\to\infty$ has a strongly convergent subsequence.

Senter and Dotson [32, p.375] introduced the concept of Condition (I) as follows.

A nonself mapping $T:K\to X$ with $F(T)\ne\emptyset$ is said to satisfy the *Condition (I)* if there exists a non-decreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$d(x,Tx)\ge f(d(x,F(T)))\quad\text{for all}\quad x\in K.$$

Using the above definitions, we obtain the following strong convergence theorem.

Theorem 4 *Let X, K, P, T and $\{x_n\}$ be the same as in Theorem 2.*

(i) *If T is semi-compact, then $\{x_n\}$ converges strongly to a fixed point of T.*
(ii) *If T satisfies Condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof

(i) It follows from (9) that $\{x_n\}$ is a bounded sequence in K. Also, by (11), we have $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Then, by the semi-compactness of T, there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $p\in K$. Moreover, by the uniform continuity of T, we have

$$d(p,Tp)=\lim_{k\to\infty} d(x_{n_k},Tx_{n_k})=0.$$

This implies that $p\in F(T)$. Again, by (9), $\lim_{n\to\infty} d(x_n,p)$ exists. Hence p is the strong limit of the sequence $\{x_n\}$. As a result, $\{x_n\}$ converges strongly to a fixed point p of T.

(ii) By virtue of (9), $\lim_{n\to\infty} d(x_n,F(T))$ exists. Further, by Condition (I) and (11), we have

$$\lim_{n\to\infty} f(d(x_n,F(T)))\le\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

That is, $\lim_{n\to\infty} f(d(x_n,F(T)))=0$. Since f is a non-decreasing function satisfying $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$, it follows that $\lim_{n\to\infty}$

$d(x_n, F(T)) = 0$. Now, Theorem 3 implies that $\{x_n\}$ converges strongly to a point p in $F(T)$. □

Remark 3

(i) Theorem 1 extends Theorem 3.2 of Saluja et al. [30] from a nearly asymptotically nonexpansive self mapping to a nearly asymptotically nonexpansive nonself mapping.

(ii) Theorem 2 extends Theorem 1 of Khan [19] from a uniformly convex Banach space to a CAT(κ) space considered in this paper.

(iii) Our results extend the corresponding results of Khan and Abbas [20] to the case of a more general class of nonexpansive mappings from a CAT(0) space to a CAT(κ) space considered in this paper.

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